L-FACTORS AND ϵ-FACTORS
SEMINAR ON THE PROOF OF LOCAL LANGLANDS

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1. Notation

- $p :=$ a fixed prime
- $K :=$ a p-adic field, i.e. a finite extension of $\mathbb{Q}_p$
- $\overline{K} :=$ an algebraic closure of K
- $O_K :=$ the ring of integers of K
- $U_n(K) :=$ the subgroup of unipotent upper triangular matrices in $\mathrm{GL}_n(K)$.

2. Generic Representations

Let us recall a few facts and definitions about generic representations.

Definition 2.1. Fix a nontrivial additive quasi-character $\psi : K \to \mathbb{C}^\times$, let $n := \max\{m \in \mathbb{N} \mid \psi(\pi_K^{-m}\mathcal{O}_K) = 1$, and define

$$U_n(K) \xrightarrow{\quad\theta_\psi\quad} K^\times$$

$$\begin{pmatrix} 1 & & u_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix} \longmapsto \psi(u_{12} + \ldots + u_{n-1,n}).$$

An irreducible and smooth representation π of $\mathrm{GL}_n(K)$ is *generic* (or *non-degenerate*) if $\mathrm{Hom}_{U_n(K)}\left(\pi|_{U_n(K)}, \theta_\psi\right)$ is nonzero.

Question 2.2. Which representations are generic?

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Fact 2.3. *(1)* π *is generic if and only if* $\pi^\vee$ *is generic.*

(2) Given any multiplicative quasicharacter $\chi : K^\times \to \mathbb{C}^\times$*,* π *is generic if and only if* $\chi\pi := (\chi \circ \det) \otimes \pi$ *is generic.*

(3) The choice of ψ *does not matter.*

Theorem 2.4 (Gelfand-Kazhdan)**.** *Every irreducible admissible supercuspidal representation is generic.*

Theorem 2.5 (Bernstein-Zelevinsky)**.** *Let* $\pi = Q(\Delta_1, \ldots, \Delta_r)$ *be irreducible and admissible for some intervals* Δ*. Then* π *is generic if and only if* Δ_i, Δ_j *are not linked for all* $i, j \in \{1, \ldots, r\}$*.*

Corollary 2.6. *Every essentially tempered (and supercuspidal) representation is generic.*

Definition 2.7. Let (π, V) be a generic representation. A *Whittaker functional* for π is a functional $\lambda : V \to \mathbb{C}$ such that $\lambda\big(\pi(u)v\big) = \theta_\psi(u)\lambda(v)$ for all $u \in U_n(K), v \in V$.

Proposition 2.8. *If* π *is generic, there exists a Whittaker functional for* π*.*

Definition 2.9. Fix a Whittaker functional λ. The *Whittaker model* for π is

$$W_{\pi,\psi} := \left\{W_v : \mathrm{GL}_n(K) \to \mathbb{C} \mid W_v(g) = \lambda\big(\pi(g)v\big)\right\}$$

with a $\mathrm{GL}_n(K)$-action given by right translation (i.e. $gW_v = W_{gv}$).

Fact 2.10. *(1)* $W_{\pi,\psi}$ *is irreducible.*

(2) $v \mapsto W_v$ *is a* $\mathrm{GL}_n(K)$*-isomorphism.*

(3) The Whittaker model $W_{\pi,\psi}$ *does not depend on choice of* ψ *(up to isomorphism).*

Then we have the *multiplicity one theorem.*

Theorem 2.11 (Shalika)**.** *The dimension of the space of Whittaker functionals is at most* 1*, i.e.*

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL}_n(K)}(\pi, \mathrm{Ind}_{U_n(K)}^{\mathrm{GL}_n(K)}\theta_\psi) \leq 1,$$

so if π *admits a Whittaker model then it is unique.*

3. $\mathrm{GL}_n(K)$ SIDE FOR GENERIC REPRESENTATIONS

Setup 3.1. Let π and π' be smooth irreducible representations of $\mathrm{GL}_n(K)$ and $\mathrm{GL}_{n'}(K)$ respectively. Assume π, π' are generic. Assume $\psi : K \to \mathbb{C}^\times$ is unitary (i.e. $\psi^{-1} = \overline{\psi}$).

3.1. Case $n' = n$.

Let

$$S(K^n) := \{\text{locally constant functions } \phi : K^n \to \mathbb{C} \text{ with compact support}\}$$

denote the Schwartz space on K^n. Let

$$Z(W, W', \phi, s) := \int_{U_n(K)\backslash \mathrm{GL}_n(K)} W(g)W'(g)\phi\left((0, \ldots, 0, 1)g\right) |\det g|^s \, dg,$$

for $W \in W_{\pi,\psi}, W' \in W_{\pi',\overline{\psi}}, \phi \in S(K^n)$, and dg is a $\mathrm{GL}_n(K)$-invariant measure on $U_n(K)\backslash \mathrm{GL}_n(K)$.

Note that $Z(W, W', \phi, s)$ absolutely converges for $\mathrm{Re}\, s >> 0$ and is a rational function in q^{-s}. In particular, the set

$$Z := \left\{Z(W, W', \phi, s) \mid W \in W_{\pi,\psi}, W' \in W_{\pi',\overline{\psi}}, \phi \in S(K^n)\right\}$$

generates a fractional ideal in $\mathbb{C}[[q^{-s}]][q^s]$ with a unique generator of the form $P(q^{-s})^{-1}$ for some polynomial $P \in \mathbb{C}[x]$ such that $P(0) = 1$.

Definition 3.2. Let $L(\pi \times \pi', s)$ be the unique generator of the fractional ideal generated by Z in $\mathbb{C}[[q^{-s}]][q^s]$.

Now we can define the ϵ-factors. Let $w_n \in \mathrm{GL}_n(K)$ be the permutation matrix corresponding to the longest Weyl group element (i.e. sending $i \mapsto n+1-i$). For $W \in \mathcal{W}_{\pi,\psi}$, define $\widetilde{W} \in \mathcal{W}_{\pi^\vee,\overline{\psi}}$ by $g \mapsto W\left(w_n^t g^{-1}\right)$ and similarly for $\widetilde{W'} \in \mathcal{W}_{\pi'^\vee,\psi}$.

Definition 3.3. Define $\epsilon(\pi \times \pi', \psi, s)$ via

$$\frac{Z(\tilde{W}, \tilde{W}', 1-s, \widehat{\phi})}{L(\pi^\vee \times \pi'^\vee, 1-s)} = \omega_{\pi'}(-1)^n \epsilon(\pi \times \pi', \psi, s) \frac{Z(W, W', s, \phi)}{L(\pi \times \pi', s)},$$

where $\omega_{\pi'} : Z\left(\mathrm{GL}_n(K)\right) \to \mathbb{C}^\times$ is the central character of (π', V') and $\widehat{\phi}$ is the Fourier transform of ϕ.

3.2. **Case $n' < n$.** For $j \in \{0, \dots, n-n'-1\}$, define

$$Z(W, W', j, s) := \int_{U_n(K) \backslash \mathrm{GL}_n(K)} \int_{M_{j \times n'}(K)} W\left(\begin{pmatrix} g & & \\ x & I_j & \\ & & I_{n-n'-j} \end{pmatrix}\right) W'(g) \left|\det g\right|^{s - \frac{n-n'}{2}} \, dx dg.$$

where dx is a Haar measure on $M_{j \times n'}(K)$ and W, W', dg as before. This converges absolutely if $\mathrm{Re}(s) >> 0$, is a rational function of q^{-s}, and generates a fractional ideal as we vary W, W', j with unique generator.

Again, let $L(\pi \times \pi', s)$ be the unique generator of the fractional ideal generated by the set of such Z. Additionally, let

$$w_{n,n'} := \begin{pmatrix} I_{n'} & \\ & w_{n-n'} \end{pmatrix} \in \mathrm{GL}_n(K).$$

Finally, define the epsilon factors as follows.

Definition 3.4. Define $\epsilon(\pi \times \pi', \psi, s)$ via

$$\frac{Z(w_{n,n'}\widetilde{W}, \widetilde{W'}, n-n'-1-j, 1-s)}{L(\pi^\vee \times \pi'^\vee, 1-s)} = w_{\pi'}(-1)^{n-1} \epsilon(\pi \times \pi', \psi, s) \frac{Z(w, w', j, s)}{L(\pi \times \pi', s)}.$$

3.3. **Case $n' > n$.** Define $L(\pi' \times \pi, s) := L(\pi \times \pi', s)$ and $\epsilon(\pi' \times \pi, \psi, s) := \epsilon(\pi' \times \pi, \psi, s)$.

3.4. **General facts.**

Fact 3.5. *(1) The L-factor does not depend on the choice of ψ.*

(2) The ϵ-factor is of the form cq^{-fs} with $c \in \mathbb{C}^\times$ and $f \in \mathbb{Z}$ which depend only on $\psi, \pi,$ and π'.

Proposition 3.6. *If π, π' are supercuspidal, then*

$$L(\pi \times \pi', s) = \prod_{\substack{\chi : K^\times \to \mathbb{C}^\times \\ \chi^{\pi'^\vee} \cong \pi}} L(\chi, s).$$

In particular, if $n' \neq n$, then $L(\pi \times \pi', s) = 1$.

We then have the following theorem.

Theorem 3.7 (Bushnell-Henniart)**.** *For π irreducible and admissible,*

$$\epsilon(\pi \times \pi^\vee, \psi, 1/2) = w_\pi(-1)^{n-1}.$$

4. $\mathrm{GL}_n(K)$ side for arbitrary smooth representations

We now inductively define these factors for more general smooth representations using the Bernstein-Zelevinsky classification.

Definition 4.1. Let π, π' be arbitrary smooth representations. Define

(1) $L(\pi \times \pi', s) = L(\pi' \times \pi, s)$ and $\epsilon(\pi \times \pi', \psi, s) = \epsilon(\pi' \times \pi, \psi, s)$

(2) If $\pi = Q(\Delta_1, \ldots, \Delta_r)$, then

$$L(\pi \times \pi', s) := \prod_{i=1}^{r} L(Q(\Delta_i) \times \pi, s)$$

$$\epsilon(\pi \times \pi', \psi, s) := \prod_{i=1}^{r} \epsilon(Q(\Delta_i) \times pi, \psi, s).$$

(3) If $\pi = Q(\Delta)$ with $\Delta = [\sigma, \sigma(r-1)]$ and $\pi' = Q(\Delta')$ with $\Delta' = [\sigma', \sigma'(r-1)]$ and $r' \geq r$, then

$$L(\pi \times \pi', s) := \prod_{i=1}^{r} L(\sigma \times \sigma', s + r + r' - i)$$

$$\epsilon(\pi \times \pi', \psi, s) := \prod_{i=1}^{r} \left(\prod_{j=1}^{r+r'-2i} \epsilon(\sigma \times \sigma', \psi, s + i + j - 1) \prod_{j=1}^{r+r'-2i-1} \frac{L(\sigma^\vee \times \sigma'^\vee, 1 - s - i - j)}{L(\sigma \times \sigma', s + i + j - 1)} \right)$$

Remark 4.2. When π, π' are arbitrary smooth representations, they may not have Whittaker models.

If we have one smooth irreducible representation π, we may also define its L-factors and ϵ-factors.

Definition 4.3. Let $\not\Vdash : K^\times \to \mathbb{C}^\times$ be the trivial multiplicative character.

$$L(\pi, s) := L(\pi \times \not\Vdash, s)$$

$$\epsilon(\pi, \psi, s) := \epsilon(\pi \times \not\Vdash, \psi, s)$$

Remark 4.4. (1) If $n = 1$, then $L(\pi, s)$ and $\epsilon(\pi, \psi, s)$ are the local factors defined in Tate's thesis.

(2) If $n > 1$ and π is supercuspidal, then $L(\pi, s) = 1$ and $\epsilon(\pi, \psi, s)$ is given by a generalized Gauss sum.

Definition 4.5. For (π, V) smooth irreducible and $t \in \mathbb{Z}_{\geq 0}$, let

$$K_n(t) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_n(\mathcal{O}_K) \mid c \in M_{1 \times (n-1)} \left(\pi_K^t \mathcal{O}_K \right) \text{ and } d \equiv 1 \pmod{\pi_K^t \mathcal{O}_K} \right\}$$

Remark 4.6. Note that $K_n(0) = \mathrm{GL}_n(\mathcal{O}_K)$.

Definition 4.7. The conductor $f(\pi)$ of π is the smallest t such that $V^{K_n(t)} \neq 0$.

Fact 4.8 (Jacquet–Piatetski-Shapiro–Shalika). *For π generic and $n(\psi)$ the exponent of ψ,*

$$\epsilon(\pi, \psi, s) = \epsilon(\pi, \psi, 0) q^{-s(f(\pi)+n\cdot n(\psi))}$$

5. Galois Side

Let $\rho\,((r, V), N)$ be a Frobenius semisimple Weil–Deligne representation. Let V_N be the kernel of N and let V^{I_K} be the invariants under the action of I_K.

Definition 5.1. For ρ Frobenius semisimple with $\Phi \in W_K$ the geometric Frobenius, define $L(\rho, s) := \det\left(1 - q^{-s}\Phi|_{V_N^{I_K}}\right)^{-1}$.

Proposition 5.2. *If ρ and ρ' are irreducible Weil–Deligne representations of dimension n and n' respectively, then*

$$L(\rho \otimes \rho', s) = \prod_{\substack{\chi : K^\times \to \mathbb{C}^\times \text{unramified} \\ \chi \otimes \rho^\vee = \rho'}} L(\chi, s).$$

Remark 5.3. If $n' \neq n$, then $L(\rho \otimes \rho', s) = 1$.

Definition 5.4. Let $\dim V = 1$, i.e. r is a character $\chi : W_K^{\text{ab}} \to \mathbb{C}^\times$.

(1) If χ is unramified,

$$\epsilon(\chi, \psi, dx) := q^{n(\psi)(1-s)} \text{vol}_{dx} \mathcal{O}_K,$$

where s is given by $\chi = |\cdot|^s$.

(2) If r is ramified, then

$$\epsilon(\chi, \psi, dx) = \int_{c^{-1}\mathcal{O}_K} r^{-1}(\text{Art}_K(x))\psi(x)\ dx,$$

where $c \in K^\times$ such that the valuation of c is $n(\psi) + f(\chi)$ where

$$f(\chi) := \min\left\{f \in \mathbb{Z}_{\geq 0} \mid \chi\left(\text{Art}_K\left(1 + \pi_K^f \mathcal{O}_K\right)\right) = 1\right\},$$

is the conductor of χ.

Theorem 5.5 (Langlands, Deligne)**.** *There is a unique function* ϵ *such that*

(1) If $\dim V = 1$*, then* $\epsilon(r, \psi, dx)$ *is as in Definition 5.4.*

(2) As a function of $\mathrm{Rep}(W_K)$*,* $\epsilon(\cdot, \psi, dx)$ *is multiplicative in exact sequences of representations of* W_K*, so we have a homomorphism*

$$\epsilon(\cdot, \psi, dx) : \mathrm{Groth}\,(\mathrm{Rep}(W_K)) \to \mathbb{C}^\times.$$

(3) If $L'/L/K$ *is a tower of finite extensions and* μ_L *and* $\mu_{L'}$ *are additive Haar measures on the Galois groups of* L *and* L' *over* K *respectively, and* $[r'] \in \mathrm{Groth}\,(\mathrm{Rep}(W_K))$ *with* $\dim[r'] = 0$*, then*

$$\epsilon\left(\mathrm{Ind}_{L'/L}[r'], \psi \circ \mathrm{tr}_{L/K,\mu_L}\right) = \epsilon\left([r'], \psi \circ \mathrm{tr}_{L'/K,\mu_{L'}}\right).$$

Remark 5.6. If $\dim V = 1$, $\epsilon(r, \psi, \alpha dx) = \alpha\epsilon(r, \psi, dx)$, hence $\epsilon(r, \psi, \alpha dx) = \alpha^{\dim[r]}\epsilon(r, \psi, dx)$. In particular, the choice of dx does not matter if $\dim[r] = 0$.

Definition 5.7. For $\rho = ((r, V), N)$ a Weil–Deligne representation of dimension n, define

$$\epsilon(\rho, \psi, s) := \epsilon(|\cdot|^s\, r, \psi, dx)\det(-\phi|_{V^{I_K}/V_N^{I_K}}),$$

where dx is the self-dual Haar measure on K with respect to the Fourier transform by ψ.

Remark 5.8. $\epsilon(\rho, \psi, s)$ is not additive in exact sequences of Weil–Deligne representations because taking coinvariants is not exact.

Definition 5.9. Let ρ be an irreducible Weil–Deligne representation of dimension n. The conductor $f(\rho)$ is given by

$$\epsilon(\rho, \psi, s) = \epsilon(\rho, \psi, 0)q^{-s(f(\rho)+n\cdot n(\psi))}.$$

Example 5.10 (Sp(m))**.** Let $m \geq 1$. Let $V = \mathbb{C}e_0 \oplus \ldots \oplus \mathbb{C}e_{n-1}$. Define Sp($m$) := $\big((r, V), N\big)$ via

$$Ne_i = e_{i+1}$$
$$Ne_{m-1} = 0$$
$$r(w)e_i = |w|^i e_i.$$

Then $V_N = \mathbb{C}e_{m-1} = V_N^{I_K}$ and $\phi e_i = q^{-i}e_i$. So

$$L(\rho, s) = 1/(1 - q^{1-s-m}).$$

Now pick $\psi : K^\times \to \mathbb{C}^\times$ such that $n(\psi) = 0$ (i.e. $\psi(\mathcal{O}_K) = 1$ but $\psi(\pi_K^{-1}\mathcal{O}_K \neq 1)$. Then

$$\epsilon(r, \psi, dx) = q^{-\frac{md}{2}} = (\mathrm{vol}_{dx}\mathcal{O}_K)^m,$$

where d is the valuation of the absolute different of K. Then

$$\epsilon(\rho, s, \psi) = (-1)^{m-1} q^{\frac{-md-(m-2)(m-1)}{2}s}.$$

Fact 5.11. *Note that* ρ *is indecomposable if and only if* $\rho \cong \rho_0 \otimes \mathrm{Sp}(m)$ *with* ρ_0 *are irreducible and* $m \geq 1$.

Consequently, ρ determines uniquely m and ρ_0 up to isomorphism. Furthermore, every Frobenius semi-simple Weil–Deligne representation is a direct sum of indecomposable Weil–Deligne representations.

6. Local Langlands Correspondence for GL$_n$ over a p-adic field

Recall for convenience,

$\mathcal{A}_n(K) :=$ {irreducible admissible representations of $\mathrm{GL}_n(K)$}

$G_n(K) :=$ {Frobenius semisimple complex Weil–Deligne representations of W_K of dimension n}.

Finally, we can state the local Langlands correspondence.

Theorem 6.1 (Local Langlands Correspondence for GL_n over a p-adic field)**.** *There is a unique collection of bijections* $\mathrm{rec}_{K,n} : \mathcal{A}_n(K) \to \mathsf{G}_n(K)$ *such that*

(1) *For* $\pi \in \mathcal{A}_1(K)$, $\mathrm{rec}_1 = \pi \circ \mathrm{Art}_K^{-1}$.

(2) *For* $\pi_1 \in \mathcal{A}_{n_1}(K), \pi_2 \in \mathcal{A}_{n_2}(K)$,

$$L(\pi_1 \times \pi_2, s) = L(\mathrm{rec}_{n_1}(\pi_1) \otimes \mathrm{rec}_{n_2}(\pi_2), s)$$

$$\epsilon(\pi_1 \times \pi_2, s, \psi) = \epsilon(\mathrm{rec}_{n_1}(\pi_1) \otimes \mathrm{rec}_{n_2}(\pi_2), s, \psi).$$

(3) *For* $\pi \in \mathcal{A}_n(K)$, $\chi \in \mathcal{A}_1(K)$,

$$\mathrm{rec}_n(\pi_\chi) = \mathrm{rec}_n(\pi) \otimes \mathrm{rec}_1(\chi).$$

(4) *For* $\pi \in \mathcal{A}_n(K)$, w_π *a central character,*

$$\det \circ \mathrm{rec}_n(\pi) = \mathrm{rec}_1(w_\pi).$$

(5) *For* $\pi \in \mathcal{A}_n(K)$,

$$\mathrm{rec}_n(\pi^\vee) = \mathrm{rec}_n(\pi)^\vee.$$

Furthermore, this collection does not depend on choice of ψ.

References

[Wed08] Torsten Wedhorn. The local Langlands correspondence for $\mathrm{GL}(n)$ over p-adic fields. In *School on Automorphic Forms on* $\mathrm{GL}(n)$, volume 21 of *ICTP Lect. Notes*, pages 237–320. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2008.

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